Back-paper Examination M. Math II Year (Differential Geometry II) 2015

Attempt all questions. Each question carries 9 marks. Books and notes maybe consulted Results proved in class, or propositions (with or without proof) from the class notes maybe used after quoting them. Results from exercises, however, must be proved in full if used.).

1. Prove that a bijective C^r-map $f : U \to V$ of open sets U, V of $\mathbb{R}^n$ which is a local C^1-diffeomorphism is a C^r- diffeomorphism.

2. Let M be a smooth compact manifold, and let $f : M \to \mathbb{R}$ be a smooth map. Prove that f is not a submersion.

3. Let $G = SL(2, \mathbb{R})$, and $A \in G$. Determine the tangent space $T_A(G)$ as a linear subspace of $\mathfrak{gl}(2, \mathbb{R})$ (the vector space of all 2×2 real matrices). If $X \in \mathfrak{sl}(2, \mathbb{R})$ (the tangent space to G at the identity I), then determine $\widetilde{X}$, the left invariant vector field corresponding to X.

4. Let M be the Moebius strip in $\mathbb{R}^3$, given as the image of the smooth map:
$$\begin{aligned} \phi : \mathbb{R} \times (-1, 1) &\to \mathbb{R}^3 \\ (s, t) &\mapsto \left(\left(2 + t\cos\frac{s}{2}\right)\cos s, \left(2 + t\cos\frac{s}{2}\right)\sin s, t\sin\frac{s}{2}\right) \end{aligned}$$
Show that there does not exist any smooth unit normal vector field on M.

5. Consider the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. For the smooth vector field X on S^2 given by $X(x, y, z) = (0, -z, y)$, compute the one parameter family of diffeomorphisms. If ω is the 1-form obtained by restricting the form dx on $\mathbb{R}^3$ to S^2, compute the Lie derivative $L_X\omega$.